1. FIRST-ORDER DIFFERENTIAL EQUATIONS

1.1 Preliminary Concepts

1. General and particular solutions: For $F(x, y, y') = 0$, any equation involving a first derivative; $y = \varphi(x)$ such that $F = 0$.

Example: $y' + y = 2 \quad \Rightarrow \quad y(x) = 2 + ce^{-x}$

$xy' = -y \quad \Rightarrow \quad y(x) = c/x$

$y' - \cos x = 0 \quad \Rightarrow \quad y(x) = \sin x + c$

2. Implicitly defined solutions

Example: $y' = -\dfrac{2xy^3 + 2}{3x^2y^2 + 8e^{4y}} \quad \Rightarrow \quad x^2y^3 + 2x + 2e^{4y} = c$

3. Integral curves: a graph of a solution

4. The initial value problem: $F(x, y, y') = 0$, initial condition: $y(x_0) = y_0$

Example: $y' = 3y, \quad y(0) = 5.7 \quad \Rightarrow \quad y(x) = 5.7e^{3x}$

5. Direction fields: $F(x, y, y') = 0 \Rightarrow y' = F(x, y)$

1.2 Separable Equations

1. Separable differential equation: $y' = A(x)B(y)$

Example: $y' = y^2 e^{-x} \quad \Rightarrow \quad y = \dfrac{1}{e^{-x} - c}$

RC circuits: Charging: $E = IR + \dfrac{Q}{C} \quad \Rightarrow \quad Q = CE(1 - e^{-t/RC})$

Discharging: $IR = \dfrac{Q}{C} \quad \Rightarrow \quad Q = Q_0 e^{-t/RC}$

1.3 Linear Differential Equations: $y'(x) + p(x)y(x) = q(x)$, integrating factor: $e^{\int p(x)dx}$

Example: $y' + y = \sin(x) \quad \Rightarrow \quad y = \dfrac{1}{2}[\sin(x) - \cos(x)] + Ce^{-x}$. $\quad y' = 3x^2 - \dfrac{y}{x}, \quad y(1) = 5$.

1.4 Exact Differential Equations

1. Potential function: For $M(x, y) + N(x, y)y' = 0$, we can find a $\varphi(x, y)$ such that $\dfrac{\partial \varphi}{\partial x} = M$ and $\dfrac{\partial \varphi}{\partial y} = N$; φ is the potential function; $M(x, y) + N(x, y)y' = 0$ is exact.

2. Exact differential equation: a potential function exists; general solution: $\varphi(x, y) = c$.

Example: $y' = -\dfrac{2xy^3 + 2}{3x^2 y^2 + 8e^{4y}}$.

3. Theorem: Test for exactness: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Example: , $x^2 + 3xy + (4xy + 2x)y' = 0$. $e^x \sin y - 2x + (e^x \cos y + 1)y' = 0$.

1.5 Integrating Factors

1. Integrating factor: $\mu(x, y) \neq 0$ such that $\mu M(x, y) + \mu N(x, y)y' = 0$ is exact.

Example: $y^2 - 6xy + (3xy - 6x^2)y' = 0$.

2. How to find integrating factor: $\dfrac{\partial(\mu M)}{\partial y} = \dfrac{\partial(\mu N)}{\partial x}$

Example: $x - xy - y' = 0$.

3. Separable equations and integrating factors: $\mu = \dfrac{1}{B}$

4. Linear equations and integrating factors: $\mu = e^{\int p(x)dx}$

1.6 Homogeneous and Bernoulli Equations

1. Homogeneous differential equation: $y' = f(\dfrac{y}{x})$; let $y = ux \Rightarrow$ separable.

Example: $xy' = \dfrac{y^2}{x} + y$.

2. Bernoulli equation: $y' + P(x)y = R(x)y^\alpha$; $\alpha = 0 \Rightarrow$ linear; $\alpha = 1 \Rightarrow$ separable; otherwise,

let $v = y^{1-\alpha} \Rightarrow$ linear

Example: $y' + \dfrac{y}{x} = 3x^2 y^3$

2. HIGHER ORDER LINEAR EVENS

2.1 Preliminary Concepts

1. $F(x, y, y', y'') = 0$, an equation that contains a second derivative, but no higher derivative.

2. Linear second-order differential equations: $R(x)y'' + P(x)y' + Q(x)y = S(x)$.

2.2 Theory of Solutions

1. The initial value problem: $y'' + p(x)y' + q(x)y = f(x); \; y(x_0) = A, \; y'(x_0) = B$.

   Example: $y'' - 12x = 0, \; y(0) = 3, \; y'(0) = -1 \quad \Rightarrow \quad y = 2x^3 - x + 3$.

2. The homogeneous linear ODEs of 2nd order: $y'' + p(x)y' + q(x)y = 0$.

3. Theorem: Let y_1 and y_2 be solutions of $y'' + p(x)y' + q(x)y = 0$ on an interval *I*. Then any linear combination of these solutions, i.e., $y = c_1 y_1 + c_2 y_2$, is also a solution.

4. Linear dependence: Two functions *f* and *g* are linearly dependent on an open interval *I* if, for some constant *c*, either $f(x) = cg(x)$ for all *x* in *I*, or $g(x) = cf(x)$ for all *x* in *I*. Linear independence: If *f* and *g* are not linearly dependent on *I*.

   Example: $y'' + y = 0 \quad \Rightarrow \quad y_1 = \cos x, \; y_2 = \sin x$.

5. Wronskian Test: Let y_1 and y_2 be solutions of $y''+p(x)y'+q(x)y=0$ on an open interval I. Then, (1) Either $W(x)=0$ for all x in I, or $W(x)\neq 0$ for all x in I. (2) y_1 and y_2 are linearly independent on I if and only if $W(x)\neq 0$ on I, where $W(x)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$.

Example: $y''+xy=0 \quad \Rightarrow \quad \begin{array}{l} y_1 = 1-\frac{1}{6}x^3+\frac{1}{180}x^6-\frac{1}{12960}x^9+\cdots, \\ y_2 = x-\frac{1}{12}x^4+\frac{1}{504}x^7-\frac{1}{45360}x^{10}+\cdots \end{array}$.

6. Theorem: Let y_1 and y_2 be linearly independent solutions of $y''+p(x)y'+q(x)y=0$ on an open interval I. Then, every solution of this differential equation on I is a linear combination of y_1 and y_2.

7. Definition: Let y_1 and y_2 be solutions of $y''+p(x)y'+q(x)y=0$ on an open interval I. (1) y_1 and y_2 form a fundamental set (or a basis) of solutions on I if y_1 and y_2 are linearly independent on I. (2) When y_1 and y_2 form a fundamental set of solutions, we call $c_1y_1+c_2y_2$, with c_1 and c_2 arbitrary constants, the general solution of the differential equation on I.

8. The nonhomogeneous equations: $y''+p(x)y'+q(x)y=f(x)$.

9. Theorem: Let y_1 and y_2 be a fundamental set of solutions of $y''+p(x)y'+q(x)y=0$ on an open interval I. Let y_p be any solution of $y''+p(x)y'+q(x)y=f(x)$ on I. Then, for any solution φ of $y''+p(x)y'+q(x)y=f(x)$, there exist numbers c_1 and c_2 such that $\varphi=c_1y_1+c_2y_2+y_p$.

2.3 Reduction of Order: Given $y''+p(x)y'+q(x)y=0$, if we know a first solution y_1, then a second solution can be the form $y_2=u(x)y_1$.

Example: $y''+4y'+4y=0,\ y_1=e^{-2x} \quad \Rightarrow \quad y_2=xe^{-2x}$.

2.4 The Constant Coefficient Homogeneous Linear Equation: $y''+Ay'+By=0$, A and B are numbers.

1. Characteristic equation: $\lambda^2+A\lambda+B=0$ obtained by substituting $y=e^{\lambda x}$ into $y''+Ay'+By=0$.

2. Case 1. $A^2-4B>0$: The general solution is $y(x)=c_1e^{ax}+c_2e^{bx}$; $a=\dfrac{-A+\sqrt{A^2-4B}}{2}$, $b=\dfrac{-A-\sqrt{A^2-4B}}{2}$.

   Example: $y''-y'-6y=0 \quad \Rightarrow \quad y=c_1e^{-2x}+c_2e^{3x}$.

3. Case 2. $A^2-4B=0$: The general solution is $y(x)=c_1e^{ax}+c_2xe^{ax}$; $a=-\dfrac{A}{2}$.

   Example: $y''-6y'+9y=0 \quad \Rightarrow \quad y=c_1e^{3x}+c_2xe^{3x}$.

4. Case 3. $A^2-4B<0$: The general solution is $y(x)=c_1e^{(p+iq)x}+c_2e^{(p-iq)x}$; $p=-\dfrac{A}{2}$, $q=\dfrac{\sqrt{4B-A^2}}{2}$.

   Example: $y''+2y'+6y=0 \quad \Rightarrow \quad y=c_1e^{(-1+\sqrt{5}i)x}+c_2e^{(-1-\sqrt{5}i)x}$.

5. An alternative general solution in the complex root case: $y(x) = e^{px}(c_1\cos(qx) + c_2\sin(qx))$.

Maclaurin expansions: $e^x = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$, $\cos x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$, $\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$.

Euler' formula: $e^{ix} = \cos x + i\sin x$.

Example: $y'' + 2y' + 6y = 0 \quad \Rightarrow \quad y = e^{-x}(c_1\cos(\sqrt{5}x) + c_2\sin(\sqrt{5}x))$.

2.5 Euler's (Euler-Cauchy) Equation: $x^2y'' + Axy' + By = 0$, let (i) $y = x^\lambda \Rightarrow$ Characteristic equation: $\lambda^2 + (A-1)\lambda + B = 0$, or (ii) let $x = e^t$, $t = \ln x$, $Y(t) = y(e^t) \Rightarrow$ $Y'' + (A-1)Y' + BY = 0$.

1. Case 1. $(A-1)^2 - 4B > 0$: The general solution is $y(x) = c_1x^a + c_2x^b$;

$$a = \frac{(1-A) + \sqrt{(A-1)^2 - 4B}}{2}, \quad b = \frac{(1-A) - \sqrt{(A-1)^2 - 4B}}{2}.$$

Example: $x^2y'' + 2xy' - 6y = 0 \quad \Rightarrow \quad y = c_1x^{-3} + c_2x^2$.

2. Case 2. $(A-1)^2 - 4B = 0$: The general solution is $y(x) = c_1x^a + c_2x^a\ln x$; $a = \frac{1-A}{2}$.

Example: $x^2y'' - 5xy' + 9y = 0 \quad \Rightarrow \quad y = c_1x^3 + c_2x^3\ln x$.

3. Case 3. $(A-1)^2 - 4B < 0$: The general solution is $y(x) = x^p(c_1\cos(q\ln x) + c_2\sin(q\ln x))$;

$$p = \frac{1-A}{2}, \quad q = \frac{\sqrt{4B - (A-1)^2}}{2}.$$

Example: $x^2y'' + 0.6xy' + 16.04y = 0 \quad \Rightarrow \quad y = x^{0.2}(c_1\cos(4\ln x) + c_2\sin(4\ln x))$.

2.6 The Nonhomogeneous Equation: $y''+p(x)y'+q(x)y=f(x)$, general solution $y=y_h+y_p$.

1. The method of variation of parameters: let $y_p=uy_1+vy_2$, then simultaneously solve

$$\begin{cases} u'y_1+v'y_2=0 \\ u'y_1'+v'y_2'=f \end{cases}.$$

Example: $y''+4y=\sec x, -\pi/4<x<\pi/4 \quad \Rightarrow$

$$y=c_1\cos 2x+c_2\sin 2x+\cos x\cos 2x+(\sin x-\frac{1}{2}\ln|\sec x+\tan x|)\sin 2x.$$

2. The method of undetermined coefficients: only applied while $p(x)$ and $q(x)$ are constant, i.e., $y''+Ay'+By=f(x)$.

Term in $r(x)$	Choice for $y_p(x)$
$ke^{\gamma x}$	$Ce^{\gamma x}$
$kx^n \ (n=0,1,\cdots)$	$K_nx^n+K_{n-1}x^{n-1}+\cdots+K_1x+K_0$
$k\cos\omega x$	$K\cos\omega x+M\sin\omega x$
$k\sin\omega x$	
$ke^{\alpha x}\cos\omega x$	$e^{\alpha x}(K\cos\omega x+M\sin\omega x)$
$ke^{\alpha x}\sin\omega x$	

Example: $y''-4y=8x^2-2x \quad \Rightarrow \quad y=c_1e^{2x}+c_2e^{-2x}-2x^2+\frac{1}{2}x-1$.

-- Modification Rule: If a term in your choice for y_p happens to be a solution of the homogeneous ODE, multiply your choice of y_p by x (or by x^2 if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE).

Example: $y'' + 2y' - 3y = 8e^{x} \quad \Rightarrow \quad y = c_1 e^{-3x} + c_2 e^{x} + 2xe^{x}$.

$$y'' - 6y' + 9y = 5e^{3x} \quad \Rightarrow \quad y = c_1 e^{3x} + c_2 xe^{3x} + \frac{5}{2}x^2 e^{3x}$$

3. The principle of superposition: $y'' + p(x)y' + q(x)y = f_1(x) + f_2(x) + \cdots + f_n(x)$, y_{pj} is a solution of $y'' + p(x)y' + q(x)y = f_j(x)$, then $y_{p1} + y_{p2} + \cdots + y_{pn}$ is a solution.

Example: $y'' + 4y = x + 2e^{-2x} \quad \Rightarrow \quad y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}(x + e^{-2x})$.

3. HIGHER ORDER LINEAR ODES

3.1 Homogeneous Linear ODEs

1. $F(x, y, y', \cdots, y^{(n)}) = 0$, a *n*th order ODE if the *n*th derivative $y^{(n)} = \dfrac{d^n y}{dy^n}$ of the unknown function $y(x)$ is the highest occurring derivative.

2. Linear ODE: $y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = g(x)$.

3. Homogeneous linear ODE: $y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$.

4. Theorem: Fundamental Theorem for the Homogeneous Linear ODE: For a homogeneous linear ODE, sums and constant multiples of solutions on some open interval *I* are again solutions on *I*. (This does not hold for a nonhomogeneous or nonlinear ODE!).

5. General solution: $y = c_1 y_1 + \cdots + c_n y_n$, where $y_1, \cdots, y_n$ is a basis (or fundamental system) of solutions on *I*; that is, these solutions are linearly independent on *I*.

6. Linear independence and dependence: *n* functions $y_1, \cdots, y_n$ are called linearly independent on some interval *I* where they are defined if the equation $k_1 y_1 + \cdots + k_n y_n = 0$ on *I* implies that all $k_1, \cdots, k_n$ are zero. These functions are called linearly dependent on *I* if this

equation also holds on *I* for some $k_1, \cdots, k_n$ not all zero.

Example: $\frac{d^4y}{dx^4} - 5\frac{d^2y}{dx^2} + 4y = 0$. Sol.: $y = c_1e^{-2x} + c_2e^{-x} + c_3e^{x} + c_4e^{2x}$.

7. Theorem: Let the homogeneous linear ODE have continuous coefficients $p_0(x), \cdots, p_{n-1}(x)$ on an open interval *I*. Then *n* solutions $y_1, \cdots, y_n$ on *I* are linearly dependent on *I* if and only if their Wronskian is zero for some $x = x_0$ in *I*. Furthermore, if *W* is zero for $x = x_0$, then *W* is identically zero on *I*. Hence if there is an x_1 in *I* at which *W* is not zero, then $y_1, \cdots, y_n$ are linearly independent on *I*, so that they form a basis of solutions of the homogeneous linear ODE on *I*.

Wronskian: $$W(y_1, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

8. Initial value problem: An ODE with n initial conditions $y(x_0) = K_0$, $y'(x_0) = K_1, \cdots,$ $y^{(n-1)}(x_0) = K_{n-1}$.

3.2 Homogeneous Linear ODEs with Constant Coefficients

1. $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = 0$: Substituting $y = e^{\lambda x}$, we obtain the characteristic equation $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$.

(i) Distinct real roots: The general solution is $y = c_1 e^{\lambda_1 x} + \cdots + c_n e^{\lambda_n x}$

Example: $y''' - 2y'' - y' + 2y = 0$. Sol.: $y = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x}$.

(ii) Simple complex roots: $\lambda = p \pm qi$, $y_1 = e^{px}\cos(qx)$, $y_2 = e^{px}\sin(qx)$.

Example: $y''' - y'' + 100y' - 100y = 0$. Sol.: $y = c_1 e^{x} + c_2 \cos 10x + c_3 \sin 10x$.

(iii) Multiple real roots: If λ is a real root of order *m*, then *m* corresponding linearly independent solutions are: $e^{\lambda x}$, $xe^{\lambda x}$, $x^2 e^{\lambda x}, \cdots$, $x^{m-1} e^{\lambda x}$.

Example: $y^{(5)} - 3y^{(4)} + 3y''' - y'' = 0$. Sol.: $y = c_1 + c_2 x + (c_3 + c_4 x + c_5 x^2) e^{x}$.

(iv) Multiple complex roots: If $\lambda = p \pm qi$ are complex double roots, the corresponding linearly independent solutions are: $e^{px}\cos(qx)$, $e^{px}\sin(qx)$, $xe^{px}\cos(qx)$, $xe^{px}\sin(qx)$.

2. Convert the higher-order differential equation to a system of first-order equations.

Example: $\dfrac{d^6 y}{dx^6} - 4\dfrac{d^4 y}{dx^4} + 2\dfrac{dy}{dx} + 15y = 0$.

3.3 Nonhomogeneous Linear ODEs

1. $y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_1(x) y' + p_0(x) y = g(x)$, the general solution is of the form:

$y = y_h + y_p$, where y_h is the homogeneous solution and y_p is a particular solution.

2. Method of undermined coefficients

Example: $y''' + 3y'' + 3y' + y = 30e^{-x}$. Sol.: $y = (c_1 + c_2 x + c_3 x^2)e^{-x} + 5x^3 e^{-x}$.

3. Method of variation of parameters: $y_p = u_1 y_1 + \cdots + u_n y_n$, where $u_k' = \dfrac{W_k}{W}$, $k = 1, \cdots, n$.

Example: $x^3 y''' - 3x^2 y'' + 6xy' - 6y = x^4 \ln x$. Sol.: $y = c_1 + c_2 x + c_3 x^2 + \dfrac{1}{6}x^4(\ln x - \dfrac{11}{6})$.

4. LAPLACE TRANSFORM

4.1 Definition and Basic Properties: initial value problem $\Rightarrow$ algebra problem $\Rightarrow$ solution of the algebra problem $\Rightarrow$ solution of the initial value problem

1. Definition (Laplace Transform): The Laplace transform $\mathcal{L}[f](s)=\int_0^{\infty} e^{-st} f(t)dt = F(s)$, for all s such that this integral converges.

Examples: $f(t)=e^{at} \Rightarrow \mathcal{L}[f](s)=\dfrac{1}{s-a}$, $s>a$. $g(t)=\sin t \Rightarrow \mathcal{L}[f](s)=\dfrac{1}{s^2+1}$.

2. Table of Laplace transform of functions

	$f(t)$	$\mathcal{L}(f)$		$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$	7	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
2	t	$1/s^2$	8	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
3	t^2	$2!/s^3$	9	$\cosh at$	$\dfrac{s}{s^2-a^2}$
4	t^n $(n=0, 1, \cdots)$	$\dfrac{n!}{s^{n+1}}$	10	$\sinh at$	$\dfrac{a}{s^2-a^2}$
5	t^a (a positive)	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	11	$e^{at}\cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
6	e^{at}	$\dfrac{1}{s-a}$	12	$e^{at}\sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$

3. Theorem (Linearity of the Laplace transform): Suppose $\mathcal{L}[f](s)$ and $\mathcal{L}[g](s)$ are defined

for $s > a$, and α and β are real numbers. Then $\mathcal{L}[\alpha f + \beta g](s) = \alpha F(s) + \beta G(s)$ for $s > a$.

4. Definition (Inverse Laplace transform): Given a function G, a function g such that $\mathcal{L}[g] = G$ is called an inverse Laplace transform of G. In this event, we write $g = \mathcal{L}^{-1}[G]$.

5. Theorem (Lerch): Let f and g be continuous on $[0, \infty)$ and suppose that $\mathcal{L}[f] = \mathcal{L}[g]$. Then $f = g$.

6. Theorem: If $\mathcal{L}^{-1}[F] = f$ and $\mathcal{L}^{-1}[G] = g$ and α and β are real numbers, then $\mathcal{L}^{-1}[\alpha F + \beta G](s) = \alpha f + \beta g$.

4.2 Solution of Initial Value Problems Using the Laplace Transform

1. Theorem (Laplace transform of a derivative): Let f be continuous on $[0, \infty)$ and suppose f' is piecewise continuous on $[0, k]$ for every positive k. Suppose also that $\lim_{k\to\infty} e^{-sk} f(k) = 0$ if $s > 0$. Then $\mathcal{L}[f'](s) = sF(s) - f(0)$.

2. Theorem (Laplace transform of a higher derivative): Suppose $f, f', \cdots, f^{n-1}$ are continuous on $[0, \infty)$ and $f^{(n)}$ is piecewise continuous on $[0, k]$ for every positive k. Suppose also that $\lim_{k\to\infty} e^{-sk} f^{(j)}(k) = 0$ for $s > 0$ and for $j = 1, 2, \cdots, n-1$. Then $\mathcal{L}[f^{(n)}](s) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$.

Examples: $y'-4y=1;\ y(0)=1 \ \Rightarrow\ y=\frac{5}{4}e^{4t}-\frac{1}{4}$.

$$y''+4y'+3y=e^{t};\ y(0)=0,\ y'(0)=2 \ \Rightarrow\ y=\frac{1}{8}e^{t}+\frac{3}{4}e^{-t}-\frac{7}{8}e^{-3t}.$$

4.3 Shifting Theorems and the Heaviside Function

1. Theorem (First shifting theorem, or shifting in the s variable): Let $\mathcal{L}[f](s)=F(s)$ for $s>b\geq 0$. Let a be any number. Then $\mathcal{L}[e^{at}f](s)=F(s-a)$ for $s>a+b$.

$\Rightarrow \mathcal{L}^{-1}[F(s-a)]=e^{at}\mathcal{L}^{-1}[F(s)]=e^{at}f(t)$.

Examples: $\mathcal{L}[e^{at}\cos(bt)] \ \Rightarrow\ \frac{s-a}{(s-a)^2+b^2}$. Find $\mathcal{L}^{-1}\left[\frac{4}{s^2+4s+20}\right] \ \Rightarrow\ e^{-2t}\sin 4t$.

2. Definition (Heaviside function): The Heaviside function (or unit step function) H is defined by $H(t)=\begin{cases}0 & \text{if } t<0\\ 1 & \text{if } t\geq 0\end{cases}$. $\Rightarrow$ $H(t-a)=\begin{cases}0 & \text{if } t<a\\ 1 & \text{if } t\geq a\end{cases}$

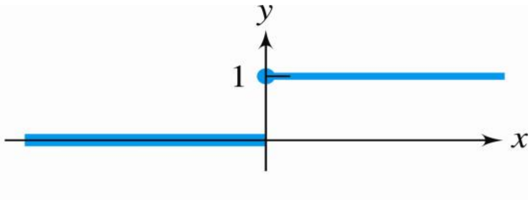


FIGURE 3.10 *The Heaviside function H(t).*

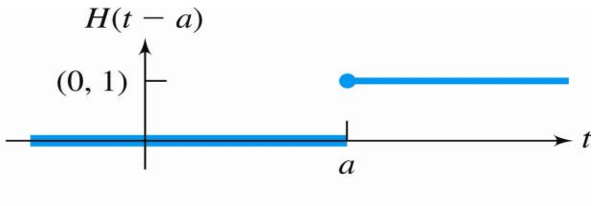
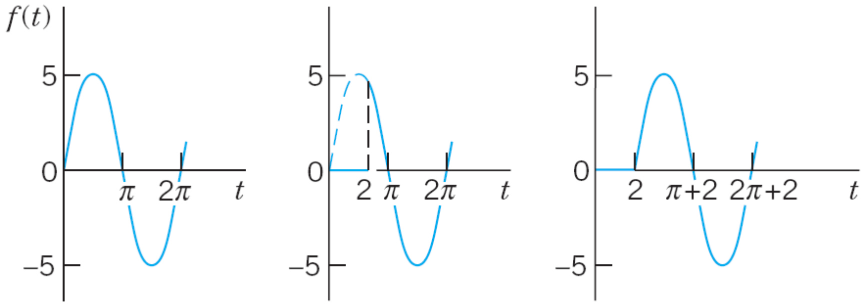


FIGURE 3.11 *A shifted Heaviside function.*

-- On-off effect: $H(t-a)g(t)=\begin{cases}0 & \text{if } t<a\\ g(t) & \text{if } t\geq a\end{cases}$, $H(t-a)g(t-a)=\begin{cases}0 & \text{if } t<a\\ g(t-a) & \text{if } t\geq a\end{cases}$



(A) $f(t) = 5 \sin t$ (B) $f(t)u(t-2)$ (C) $f(t-2)u(t-2)$

3. Definition (Pulse). A pulse is a function of the form $H(t-a)-H(t-b)$, in which $a<b$.

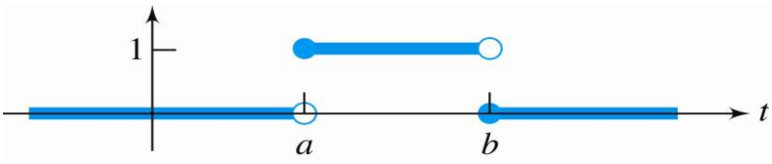


FIGURE 3.13 *Pulse function* $H(t-a)-H(t-b)$.

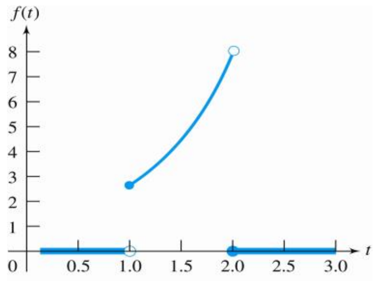


FIGURE 3.14 *Graph of* $f(t)=[H(t-1)-H(t-2)]e^t$.

4. Theorem (Second shifting theorem, or shifting in the t variable): Let $\mathcal{L}[f](s)=F(s)$ for $s>b$. Then $\mathcal{L}[H(t-a)f(t-a)](s)=e^{-as}F(s)$ for $s>b$. $\Rightarrow$

$$\mathcal{L}^{-1}[e^{-as}F(s)]=H(t-a)f(t-a)$$

Examples: $\mathcal{L}[H(t-a)] \Rightarrow \dfrac{e^{-as}}{s}$. $\mathcal{L}^{-1}[\dfrac{se^{-3s}}{s^2+4}] \Rightarrow H(t-3)xos(2(t-3))$.

Compute $\mathcal{L}[g]$, where $g(t)=0$ for $0 \le t < 2$ and $g(t)=t^2+1$ for $t \ge 2$.

$$\Rightarrow e^{-2s}\left[\frac{2}{s^3}+\frac{4}{s^2}+\frac{5}{s}\right].$$

Solve $y''+4y=f(t)$, $y(0)=y'(0)=0$, $f(t)=\begin{cases} 0 & \text{if } t<3 \\ t & \text{if } t \ge 3 \end{cases}$. $\Rightarrow$

$$y=H(t-3)\left[\frac{3}{4}+\frac{1}{4}(t-3)-\frac{3}{4}\cos(2(t-3))-\frac{1}{8}\sin(2(t-3))\right].$$

4.4 Convolution

1. Definition (Convolution): If f and g are defined on $[0,\infty)$, then the convolution $f*g$ of f with g is the function defined by $(f*g)(t)=\int_0^t f(t-\tau)g(\tau)d\tau$ for $t\geq 0$.

2. Theorem (Convolution theorem): If $f*g$ is defined, then

$$\mathcal{L}[f*g]=\mathcal{L}[f]\mathcal{L}[g]=F(s)G(s).$$

3. Theorem: Let $\mathcal{L}^{-1}[F]=f$ and $\mathcal{L}^{-1}[G]=g$. Then $\mathcal{L}^{-1}[FG]=f*g$.

Example: $\mathcal{L}^{-1}\left[\frac{1}{s(s-4)^2}\right] \Rightarrow \frac{1}{4}te^{4t}-\frac{1}{16}e^{4t}+\frac{1}{16}$.

Determine f such that $f(t)=2t^2+\int_0^t f(t-\tau)e^{-\tau}d\tau \Rightarrow 2t^2+\frac{2}{3}t^3$.

4. Theorem: If $f*g$ is defined, so is $g*f$, and $f*g=g*f$.

Example: Solve $y''-2y'-8y=f(t);\quad y(0)=1, y'(0)=0 \Rightarrow$

$$\frac{1}{6}f*e^{4t}-\frac{1}{6}f*e^{-2t}+\frac{1}{3}e^{4t}+\frac{2}{3}e^{-2t}.$$

4.5 Unit Impulses and the Dirac's Delta Function

1. Dirac's delta function: $\delta(t)=\lim_{\varepsilon\to 0+}\delta_\varepsilon(t)$, where $\delta_\varepsilon(t)=\frac{1}{\varepsilon}[H(t)-H(t-\varepsilon)]$;

$$\mathcal{L}[\delta(t-a)]=e^{-as};\quad \mathcal{L}[\delta(t)]=1.$$

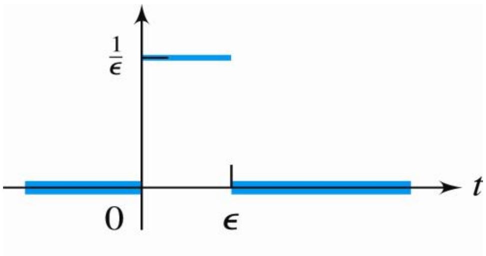


FIGURE 3.31 *Graph of* $\delta_\epsilon(t-a)$.

2. Theorem (Filtering property): Let $a > 0$ and let f be integrable on $[0,\infty)$ and continuous at a. Then $\int_0^\infty f(t)\delta(t-a)dt = f(a)$

-- Let $f(t) = e^{-st} \Rightarrow \int_0^\infty f(t)\delta(t-a)dt = \int_0^\infty e^{-st}\delta(t-a)dt = e^{-sa} = f(a) \Rightarrow$ the definition of the Laplace transformation of the delta function.

Example: Solve $y''+2y'+2y = \delta(t-3); \quad y(0) = y'(0) = 0 \Rightarrow y = H(t-3)e^{-(t-3)}\sin(t-3)$.

4.6 Laplace Transform Solution of Systems

Example: Solve the system:

$$\begin{aligned} &x''-2x'+3y'+2y = 4,\\ &2y'-x'+3y = 0,\\ &x(0) = x'(0) = y(0) = 0. \end{aligned} \Rightarrow \begin{aligned} x &= -\frac{7}{2} - 3t + \frac{1}{6}e^{-2t} + \frac{10}{3}e^{t}\\ y &= -1 + \frac{1}{3}e^{-2t} + \frac{2}{3}e^{t} \end{aligned}$$

4.7 Differential Equations with Polynomial Coefficients

1. Theorem: Let $\mathcal{L}[f](s) = F(s)$ for $s > b$ and suppose that F is differentiable. Then $\mathcal{L}[tf(t)](s) = -F'(s)$ for $s > b$.

2. Corollary: Let $\mathcal{L}[f](s) = F(s)$ for $s > b$ and let n be a positive integer. Suppose F is n times differentiable. Then $\mathcal{L}[t^n f(t)](s) = (-1)^n \frac{d^n}{ds^n}F(s)$ for $s > b$.

Example: $ty''+(4t-2)y'-4y=0; \quad y(0)=1 \;\Rightarrow$

$$y=e^{-4t}+2te^{-4t}+c\left[-\frac{1}{32}+\frac{1}{16}t+\frac{1}{32}e^{-4t}+\frac{1}{16}te^{-4t}\right].$$

3. Theorem: Let f be piecewise continuous on $[0,k]$ for every positive number k and suppose there are numbers M and b such that $|f(t)|\le Me^{bt}$ for $t\ge 0$. Let $\mathcal{L}[f](s)=F(s)$. Then $\lim\limits_{s\to\infty}F(s)=0$.

Example: $y''+2ty'-4y=1; \quad y(0)=y'(0)=0 \;\Rightarrow\; y=\frac{1}{2}t^2$.

5. SERIES SOLUTIONS

5.1 Power Series Solutions of Initial Value Problems

1. Definition (Analytic function): A function *f* is analytical at x_0 if $f(x)$ has a power series representation in some open interval about x_0: $f(x)=\sum_{n=0}^{\infty} a_n (x-x_0)^n$ in some interval (x_0-h, x_0+h).

Example: Taylor series: $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$, $a_n=\frac{f^{(n)}(x_0)}{n!}$.

Maclaurin series: $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$, i.e., $x_0=0$ in Taylor series.

$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$ at $x=0$.

2. Theorem: Let *p* and *q* be analytic at x_0. Then the initial value problem $y'+p(x)y=q(x)$; $y(x_0)=y_0$ has a solution that is analytical at x_0.

Example: $y'+e^x y=x^2$; $y(0)=4 \Rightarrow y(x)=4-4x+x^3+\frac{x^4}{12}+\cdots$

3. Theorem: Let *p*, *q* and *f* be analytic at x_0. Then the initial value problem $y''+p(x)y'+q(x)y=f(x)$; $y(x_0)=A$, $y'(x_0)=B$ has a unique solution that is also analytical at x_0.

Examples: $y''-xy'+e^x y=4$; $y(0)=1$, $y'(0)=4 \;\Rightarrow\; y(x)=1+4x+\frac{3}{2}x^2-\frac{x^3}{6}+\cdots$.

$y''+\cos(x)y'+4y=2x-1 \;\Rightarrow$

$$y(x)=a+bx+\frac{-1-4a-b}{2}x^2+\frac{4a-3b+3}{6}x^3+\cdots,\quad a=y(0),\quad b=y'(0).$$

5.2 Power Series Solutions Using Recurrence Relations

1. Coefficients developed to be a recurrence relation

Example: $y''+x^2y=0$ at $x=0 \;\Rightarrow\; a_2=0$, $a_3=0$, $a_n=-\frac{1}{n(n-1)}a_{n-4}$, $n=4,5,\cdots$,

$$y=a_0\left(1-\frac{1}{12}x^4+\frac{1}{672}x^8+\cdots\right)+a_1\left(x-\frac{1}{20}x^5+\frac{1}{1440}x^9+\cdots\right).$$

2. Two-term recurrence relation

Example: $y''+x^2y'+4y=1-x^2$ at $x=0 \;\Rightarrow\; a_2=\frac{1}{2}-2a_0$, $a_3=-\frac{2}{3}a_1$,

$$a_4=-\frac{1}{4}+\frac{2}{3}a_0-\frac{1}{12}a_1,\quad a_n=-\frac{4a_{n-2}+(n-3)a_{n-3}}{n(n-1)},\quad n=5,6,\cdots.$$

5.3 Singular Points and the Method of Frobenius

1. Definition (Ordinary and singular points): x_0 is an ordinary point of equation $P(x)y''+Q(x)y'+R(x)y=F(x)$ if $P(x_0)\neq 0$ and $Q(x)/P(x)$, $R(x)/P(x)$, and

$F(x)/P(x)$ are analytic at x_0. x_0 is a singular point of equation $P(x)y''+Q(x)y'+R(x)y=F(x)$ if x_0 is not an ordinary point.

Example: $x^3(x-2)^2y''+5(x+2)(x-2)y'+3x^2y=0 \Rightarrow x=0,\ x=2$ are singular points.

2. Definition (Regular and irregular singular points): x_0 is a regular singular point of $P(x)y''+Q(x)y'+R(x)y=0$ if x_0 is a singular point, and the functions $(x-x_0)\dfrac{Q(x)}{P(x)}$ and $(x-x_0)^2\dfrac{R(x)}{P(x)}$ are analytic at x_0. A singular point that is not regular is said to be an irregular singular point.

Example: $x^3(x-2)^2y''+5(x+2)(x-2)y'+3x^2y=0 \Rightarrow x=0$ is an irregular singular point, $x=2$ is a regular singular points.

3. Frobenius series: $y(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^{n+r}$.

4. Theorem (Method of Frobenius): Suppose x_0 is a regular singular point of $P(x)y''+Q(x)y'+R(x)y=0$. Then there exists at least one Frobenius solution $y(x)=\sum_{n=0}^{\infty}c_n(x-x_0)^{n+r}$ with $c_0\neq 0$. Further, if the Taylor expansions of $(x-x_0)\dfrac{Q(x)}{P(x)}$ and $(x-x_0)^2\dfrac{R(x)}{P(x)}$ about x_0 converge in an open interval (x_0-h,x_0+h), then this Frobenius series also converges in this interval, except

perhaps at x_0 itself.

There will be an indicial equation used to determine the values of r.

Example: $x^2 y''+x(2x+\frac{1}{2})y'+(x-\frac{1}{2})y=0 \quad \Rightarrow \quad r=1: \; c_n=-\frac{2n+1}{n(n+\frac{3}{2})}c_{n-1}\,, \quad n=1, 2, \cdots;$

$r=-\frac{1}{2}: \; c_n^*=-\frac{2n-2}{n(n-\frac{3}{2})}c_{n-1}^*, \quad n=1, 2, \cdots$

5.4 Second Solutions and Logarithm Factors

1. Theorem (A second solution in the method of Frobenius): Suppose 0 is a regular singular point of $P(x)y''+Q(x)y'+R(x)y=0$. Let r_1 and r_2 be roots of the indicial equation. If these are real, suppose $r_1 \ge r_2$. Then (a) If r_1-r_2 is not an integer, there are two linearly independent Frobenius solutions: $y_1(x)=\sum_{n=0}^{\infty} c_n x^{n+r_1}$ and $y_2(x)=\sum_{n=0}^{\infty} c_n^* x^{n+r_2}$, with $c_0 \ne 0$ and $c_0^* \ne 0$. These solutions are valid in some interval $(0, h)$ or $(-h, 0)$. (b) If $r_1-r_2=0$, there is a Frobenius solution $y_1(x)=\sum_{n=0}^{\infty} c_n x^{n+r_1}$ with $c_0 \ne 0$ as well as a second solution: $y_2(x)=y_1(x)\ln x+\sum_{n=1}^{\infty} c_n^* x^{n+r_1}$. Further, y_1 and y_2 form a fundamental set of solutions on some interval $(0, h)$. (c) If r_1-r_2 is a positive integer, then there

is a Frobenius solutions: $y_1(x)=\sum_{n=0}^{\infty}c_n x^{n+r_1}$. In this case there is a second solution of the form $y_2(x)=ky_1(x)\ln x+\sum_{n=0}^{\infty}c_n^* x^{n+r_2}$. If $k=0$ this is a second Frobenius series solution; if not, the solution contains a logarithm term. In either event, y_1 and y_2 form a fundamental set on some interval $(0,h)$.

Examples: $x^2y''+5xy'+(x+4)y=0 \Rightarrow$ $r=-2$: $y_1(x)=c_0\sum_{n=0}^{\infty}(-1)^n\frac{1}{(n!)^2}x^{n-2}$, $c_1^*=2$,

$$c_n^*=-\frac{1}{n^2}c_{n-1}^*-\frac{2(-1)^n}{n(n!)^2},\quad n=2,3,\cdots.$$

$x^2y''+x^2y'-2y=0 \Rightarrow$ $r=2$ or $r=-1$.

6. FOURIER SERIES

6.1 The Fourier Series of a Function

1. $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos(\frac{n\pi x}{L})+b_n\sin(\frac{n\pi x}{L})$, $-L\le x\le L$, $\int_{-L}^{L}f(x)dx$ exists.

2. Lemma 13.1: If n and m are nonnegative integers, i. $\int_{-L}^{L}\cos(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})dx=0$;

   ii. $\int_{-L}^{L}\cos(\frac{n\pi x}{L})\cos(\frac{m\pi x}{L})dx=\int_{-L}^{L}\sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})dx=0$, if $n\ne m$;

   iii. $\int_{-L}^{L}\cos^2(\frac{n\pi x}{L})dx=\int_{-L}^{L}\sin^2(\frac{n\pi x}{L})dx=L$, if $n\ne 0$.

3. Definition 13.1: Let *f* be a Riemann integrable function on $[-L, L]$, then Fourier series of *f* on $[-L, L]$: $\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos(\frac{n\pi x}{L})+b_n\sin(\frac{n\pi x}{L})$; Fourier coefficients of *f* on $[-L, L]$: $a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos(\frac{n\pi x}{L})dx$, $b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin(\frac{n\pi x}{L})dx$ for $n=0, 1, 2, \cdots$.

4. Definition 13.2: Even and odd functions:

   f is an even function on $[-L, L]$ if $f(-x)=f(x)$ for $-L\le x\le L$; *f* is an odd function on $[-L, L]$ if $f(-x)=-f(x)$ for $-L\le x\le L$; $\text{even}\cdot\text{even}=\text{even}$; $\text{odd}\cdot\text{odd}=\text{even}$; $\text{even}\cdot\text{odd}=\text{odd}$.

$\int_{-L}^{L} f(x)dx = 0$ if *f* is odd on $[-L, L]$; $\int_{-L}^{L} f(x)dx = 2\int_{0}^{L} f(x)dx$ if *f* is even on $[-L, L]$.

6.2 Convergence of Fourier Series

1. Definition 13.3: Piecewise continuous function

   f is piecewise continuous on [*a*, *b*] if

   1. *f* is continuous on [*a*, *b*] except perhaps at finitely many points.

   2. Both $\lim_{x\to a+} f(x)$ and $\lim_{x\to b-} f(x)$ exist and are finite.

   3. If x_0 is in (*a*, *b*) and *f* is not continuous at x_0, then $\lim_{x\to x_0+} f(x)$ and $\lim_{x\to x_0-} f(x)$ exist and are finite.

2. Definition 13.4: Piecewise smooth function

   f is piecewise smooth on [*a*, *b*] if *f* and f' are piecewise continuous on [*a*, *b*].

3. Theorem 13.1: Convergence of Fourier series

   Let *f* is piecewise smooth on $[-L, L]$. Then for $-L < x < L$, the Fourier series of *f* on $[-L, L]$ converge to $\frac{1}{2}(f(x+) + f(x-))$.

4. Convergence at the endpoints

5. Definition 13.5: Right derivative $f_R'(c) = \lim_{h\to 0+} \dfrac{f(c+h) - f(c+)}{h}$

6. Definition 13.6: Left derivative $f_L'(c) = \lim_{h \to 0-} \frac{f(c+h) - f(c-)}{h}$

7. Theorem 13.2: Let *f* is piecewise smooth on $[-L, L]$. Then,

   i. If $-L < x < L$ and *f* has a left and right derivative at *x*, then the Fourier series of *f* on $[-L, L]$ converge at *x* to $\frac{1}{2}(f(x+) + f(x-))$.

   ii. If $f_R'(-L)$ and $f_L'(L)$ exist, then at both L and $-L$, the Fourier series of *f* on $[-L, L]$ converge to $\frac{1}{2}(f(-L+) + f(L-))$.

8. Partial sums of Fourier series

6.3 Fourier Cosine and Sine Series

1. The Fourier cosine series of a function

   Let *f* be integrable on the half-interval [0, *L*]: $f_e(x) = \begin{cases} f(x), & \text{for } 0 \le x \le L \\ f(-x), & \text{for } -L \le x < 0 \end{cases}$, f_e is an even function and called even extension of *f* on $[-L, L]$.

   Fourier cosine series of *f* on [0, *L*]: $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L})$; Fourier cosine coefficients of *f* on [0, *L*]: $a_n = \frac{2}{L}\int_0^L f_e(x)\cos(\frac{n\pi x}{L})dx = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})dx$.

2. Theorem 13.3: Convergence of Fourier cosine series

Let *f* is piecewise continuous on $[0, L]$. Then,

i. If $0 < x < L$ and *f* has a left and right derivative at *x*, then the Fourier cosine series of *f* on $[0, L]$ converges at *x* to $\frac{1}{2}(f(x+) + f(x-))$.

ii. If *f* has a right derivative at 0, then the Fourier cosine series of *f* on $[0, L]$ converges at 0 to $f(0+)$.

iii. If *f* has a left derivative at *L*, then the Fourier cosine series of *f* on $[0, L]$ converges at *L* to $f(L-)$.

3. The Fourier sine series of a function

Let *f* be integrable on the half-interval [0, *L*]: $f_o(x) = \begin{cases} f(x), & \text{for } 0 \le x \le L \\ -f(-x), & \text{for } -L \le x < 0 \end{cases}$, f_o is an odd function and called odd extension of *f* on $[-L, L]$.

Fourier sine series of *f* on [0, *L*]: $\sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})$; Fourier sine coefficients of *f* on [0, *L*]:

$$b_n = \frac{2}{L}\int_0^L f_o(x)\sin(\frac{n\pi x}{L})dx = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx.$$

4. Theorem 13.4: Convergence of Fourier sine series

Let *f* is piecewise continuous on $[0, L]$. Then, If $0 < x < L$ and *f* has a left and right derivative at *x*, then the Fourier sine series of *f* on $[0, L]$ converges at *x* to $\frac{1}{2}(f(x+) + f(x-))$.

ii. At 0 and L, the Fourier sine series of *f* on $[0, L]$ converges to 0.

6.4 Integration and Differentiation of Fourier Series

1. Theorem 13.5: Integration of Fourier series

Let *f* be piecewise continuous on $[-L, L]$, with Fourier series

$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})$. Then, for any *x* on $[-L, L]$,

$$\int_{-L}^{x} f(t)dt = \frac{1}{2}a_0(x+L) + \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[a_n \sin(\frac{n\pi x}{L}) - b_n\left(\cos(\frac{n\pi x}{L}) - (-1)^n\right)\right].$$

2. Theorem 13.6: Differentiation of Fourier series

Let *f* be continuous on $[-L, L]$ and suppose $f(L) = f(-L)$. Let f' be piecewise continuous on $[-L, L]$. Then, $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})$, and at each point in $(-L, L)$ where $f''(x)$ exists, $f'(x) = \sum_{n=1}^{\infty}\frac{n\pi}{L}\left[-a_n \sin(\frac{n\pi x}{L}) + b_n \cos(\frac{n\pi x}{L})\right]$.

3. Theorem 13.7: Bessel's inequalities: i. The coefficients in the Fourier sine expansion of *f* on $[0, L]$ satisfy $\sum_{n=1}^{\infty} b_n^2 \le \frac{2}{L}\int_0^L f^2(x)dx$; ii. The coefficients in the Fourier cosine expansion

of f on $[0, L]$ satisfy $\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} a_n^2 \le \frac{2}{L}\int_0^L f^2(x)dx$; iii. If f is integrable on $[-L, L]$, then the Fourier coefficients of f on $[-L, L]$ satisfy $\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \le \frac{1}{L}\int_{-L}^{L} f^2(x)dx$

4. Theorem 13.8: Uniform and absolute convergence of Fourier series: Let f be continuous on $[-L, L]$ and let f' be piecewise continuous. Suppose $f(-L) = f(L)$. Then, the Fourier series of f on $[-L, L]$ converges absolutely and uniformly to $f(x)$ on $[-L, L]$.

5. Theorem 13.9: Parseval's theorem: Let f be continuous on $[-L, L]$ and let f' be piecewise continuous. Suppose $f(-L) = f(L)$. Then, the Fourier coefficients of f on $[-L, L]$ satisfy $\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{L}\int_{-L}^{L} f^2(x)dx$

6.5 The Phase Angle Form of a Fourier Series

1. Periodic, fundamental period

2. Definition 13.7: Phase angle form: Let f have fundamental period p. Then the phase angle form of the Fourier series of f is $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 x + \delta_n)$, in which $\omega_0 = \frac{2\pi}{p}$, $c_n = \sqrt{a_n^2 + b_n^2}$, and $\delta_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)$ for $n = 1, 2, \cdots$.

Harmonic form, nth harmonic, harmonic amplitude, phase angle.

6.6 Complex Fourier Series and the Frequency Spectrum

1. Conjugate, magnitude, argument, polar form.

2. Definition 13.8: Complex Fourier series: Let *f* have fundamental period *p*. Let $\omega_0 = \frac{2\pi}{p}$.

Then the complex Fourier series of *f* is $\sum_{n=-\infty}^{\infty} d_n e^{in\omega_0 x}$, where $d_n = \frac{1}{p}\int_{-p/2}^{p/2} f(t)e^{-in\omega_0 t}\,dt$ for $n = 0, \pm 1, \pm 2, \cdots$. The numbers d_n are the complex Fourier coefficients of *f*.

3. Theorem 13.10: Let *f* be periodic with fundamental period *p*. Let *f* be piecewise smooth on $[-p/2,\ p/2]$. Then at each *x* the complex Fourier series converges to $\frac{1}{2}(f(x+) + f(x-))$.